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DEFLECTION ANALYSIS OF REINFORCED CONCRETE SLAB USING FINITE DIFFERENCE METHOD

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ABSTRACT:

The performance of reinforced concrete (RC) slabs under service and dead loads is a measure of their ability to support the weight of the structure. The value of deflection is significant for serviceability. This paper presents a numerical simulation model for RC slab deflection based on the finite difference method (FDM) where the governing fourth-order partial differential equation (the deflection equation) was converted to a set of algebraic equations. A MATLAB program code was developed to solve the system of ordinary algebraic equations by applying the boundary condition at the supports of the slab. Twelve slabs were analysed for deflection, under uniform ($5\ KN/m^2$) load under triangle load ($10\ KN/m^2$). The slab dimensions vary between 4×4 , 6×6 , and $8 \times 8\ m$, The validation of the FDM model was verified with the results of an analytical solution and Ansys software for the present slabs. Whereas the accuracy and reliability of the MATLAB code are studied in terms of convergence analysis and mesh sensitivity. The results showed that the 2-D FDM model of the concrete slab agreed with the Ansys data and the analytical solution. However, the code required a significant large number of nodes to match the exact solution.

تحليل هطول البلاطة الخرسانية باستخدام طريقة الفروق المحدودة

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الكلمات المفتاحية:

الهطول

بلاطات خرسانية مسلحة

طريقة الفروق المحدودة

طرق عديدة

معادلة تفاضلية من الدرجة الرابعة

المستخلص:

يعد أداء البلاطات الخرسانية المسلحة (**RC**) تحت الخدمة والأحمال الميتة مقياسًا لقدرتها على دعم وزن المنشى. ومع ذلك، فإن ضمان الأداء الوظيفي والمتانة والراحة لمستخدمي المبنى يُعرف بحالة حد إمكانية الخدمة. قيمة الهطول مهمة لقابلية الخدمة. يقدم هذا البحث نموذج محاكاة عددية لهطول بلاطة **RC** بالاعتماد على طريقة الفرق المحدود (**FDM**) حيث تم تحويل المعادلة التفاضلية الجزئية الحاكمة من الدرجة الرابعة (معادلة الهطول) إلى مجموعة من المعادلات الجبرية. تم تطوير كود برنامج **MATLAB** لحل نظام المعادلات الجبرية العادية من خلال تطبيق الشرط الحدودي على دعامات اللوحة. تم تحليل اثني عشر بلاطة من أجل الانحراف، نصفها تحت حمل موحد (5 كيلو نيوتن / م ^ 2) بينما النصف الآخر تحت حمل مثلث (10 كيلو نيوتن / م ^ 2)، بالإضافة إلى أن أبعاد البلاطات تتراوح بين 4 × 4 و 6×6 و 8×8 م، تم التحقق من صحة نموذج **FDM** من خلال نتائج القيمة الحقيقية للهطول و من برنامج **Ansys** للبلاطات الحالية. حيث أنه تمت دراسة دقة وموثوقية كود ماتلاب من حيث تحليل التقارب وحساسية الشبكة. أظهرت النتائج أن نموذج **FDM** ثنائي الأبعاد للبلاطة الخرسانية يتوافق مع بيانات **Ansys** و مع القيمة الحقيقية. لكن يتطلب الكود عددًا كبيرًا من العقد لمطابقة الحل الدقيق.

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INTRODUCTION

Due to the advanced design methods along with the use of high-strength materials structure members with relatively small cross sections and higher slender ratios may be used in beams, columns, and slabs with sufficient capacity to resist the ultimate loads. However, serviceability requirements such as deflection have become a real problem and in many cases, deflection controls the design process. To achieve the requirements of serviceability, the calculation of deflection of RC slabs is unpopular with designers due to the complexities involved in deflection analysis [1]. therefore, design code and standards for reinforced concrete slabs provide a simplified and approximated methods to calculate the deflection for certain values [2], [3]. Nonetheless, these methods are for simplified cases and give overestimated values. High values of deflection in slabs result in vibration problems and discomfort of use in the building even though the building is still safe [4].

As in many engineering applications, the problem is modeled mathematically using partial differential equations (PDEs). The deflection phenomenon is not an exception, whereas the governing equation for deflection is a fourth-order PDE in two dimensions. The analytic solutions for fourth order PDEs are not available in most cases therefore, a wide variety of PDEs have been solved using numerical methods. One of the most famous of these methods is the Finite Element Method (FEM) [5] which is used in most analysis software like Ansys. Another well-known numerical method is the Finite Difference Method (FDM) [6] which is adopted in the present work. The general procedure of these methods involves converting the partial differential equations to a set of algebraic equations then the system of equations is solved to obtain the deflection at each point [7].

The present paper provides the evaluation of deflection for simply supported reinforced concrete slab using the analytic solutions for fourth-order PDE and using the FEM by Ansys software and also by using FDM where a MATLAB program code was developed to calculate the deflection within the RC slab to investigate the accuracy and reliability for the MATLAB code comparing with the Ansys and analytic solutions. The first section of this paper presents an introduction to slab deflection and key principles related to plate theory, followed by a brief review of research on deflection and finite element method (FDM), the second section discusses the explanation of FDM and mathematical formulation of the deflection equation, third section explains the description of the concrete slab model and boundary conditions. The following section shows the results and discussion to introduce the study's outcomes. The last section includes the paper's conclusion.

The typical span for a reinforced concrete slab is about 20 ft (6.1 m) in residential and industrial construction. However, influenced by advanced architectural designs, longer spans have become more demanding to acquire in structural systems. Many factors contributed to this evolution such as using the strength-design method rather working-stress design method which allowed more slender sections and the use of high-strength material in both steel and concrete [8].

A solid slab supported from all four sides was the original slab system for reinforced concrete. In this system, if the long span is twice the short span or more, the panel acts as a one-way slab and if the ratio between long span to the short span approaches unity (square panel) significant load is transferred by bending in both directions thus, the plate acts as two-way slab.

A. Johari and Z. Delavar [9] published a paper that demonstrate deflection analysis of RC slab using MATLAB code by applying the Direct Design Method (DDM), the main objective of the study was the reliability indices of the deflection of a square slab. The results showed that analysis using (DDM) were classified as Good and Above average. In addition, the critical parameter resulted from the study were the thickness of the slab and floor type.

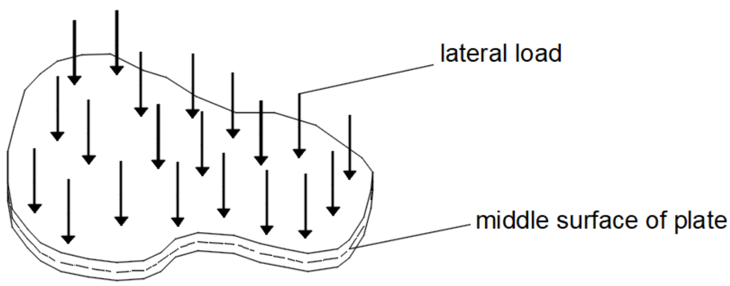

Figure (1): A schematic diagram of plate.

Another research paper proposed by O. Sucharda and J. Kubosek [10] demonstrates the analysis of RC slab using both FDM and FEM and comparing the results of the two. The FEM was studied by Scia Engineer program and the FDM was developed using MATLAB Algorithm. The paper discusses the deflection of four slabs with different dimensions, loads, and support types to estimate the accuracy of the two methods. The maximum deflection calculated using the FDM showed good conversion to the maximum deflection calculated by using the FED.

THEORETICAL BACKGROUND

Finite Difference Method is a numerical approach used to solve differential and partial differential equations as a grid form. Its widely used for solving engineering and physics problems. The basic principle of this method is to discretize the slab into set of nodes. These set of nodes represent a system of linear equations. Solving of these linear equations will give the value of deflection at each node [10].

Plate is a thin structural element with a very small thickness compared to its other dimensions. The plates are flat surfaces they resist the load primarily by bending in both directions. The plates are very common structural elements used in Civil, Mechanical, Aerospace and Marine Engineering. some applications of plates are in floor slab, bridge deck slab, foundation slab, base plate, walls and many other cases.

The plate theory can be classified into three categories as follows:

I. Thin plate with small deflection: deflection of the plate is small in comparison to its thickness. Bending moments and twisting moments are produced. These types of plates are used to model CR slabs.

II. Thin plate with large deflection: in this case, deflections are not small compared to the plate thickness, with nonlinear geometry is taken in account.

III. Thick plate: in thick plates, thickness of the plate is greater than 1/10th of its longer dimension and therefore shear deformation contributes to the deflection.

The governing differential equation for the deflection of thin plate i.e.(RC Slab) under pure bending is based on the biharmonic equation shown below:

$$D\nabla^4 z = q(x,y) \qquad (1)$$

Where ∇^4 is the Biharmonic Operator which is giving by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (2)$$

Thus, the small deflection at any point is giving by

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{q(x,y)}{D} \qquad (3)$$

Where:

z = small deflection

q = Applied transverse load

D = Flexural rigidity of the plate = $\frac{Eh^3}{12(1-v^2)}$

Numerical solution using the Finite Different Method depends on the approximation of Taylor derivatives for differential equations as follows, for 4th order derivative in x-direction.

$$\frac{\partial^4 z_i}{\partial x^4} \approx \frac{z_{i+2} - 4z_{i+1} + 6z_i - 4z_{i-1} + z_{i-2}}{\Delta_x^4} \qquad (4)$$

Similarly, in y-direction

$$\frac{\partial^4 z_i}{\partial y^4} \approx \frac{z_{j+2} - 4z_{j+1} + 6z_j - 4z_{j-1} + z_{j-2}}{\Delta_y^4} \qquad (5)$$

Also, for the for the second derivative in both x & y directions

$$\frac{\partial^4 z_{(x,y)}}{\partial x^2 \partial y^2} \approx \{z_{(i-1,j+1)} - 2z_{(i,j+1)} + z_{(i+1,j+1)} - 2z_{(i-1,j)} + 4z_{(i,j)} - 2z_{(i+1,j)} + z_{(i-1,j-1)} - 2z_{(i,j-1)} + z_{(i+1,j-i)}\}/\Delta_y^2\Delta_x^2 \qquad (6)$$

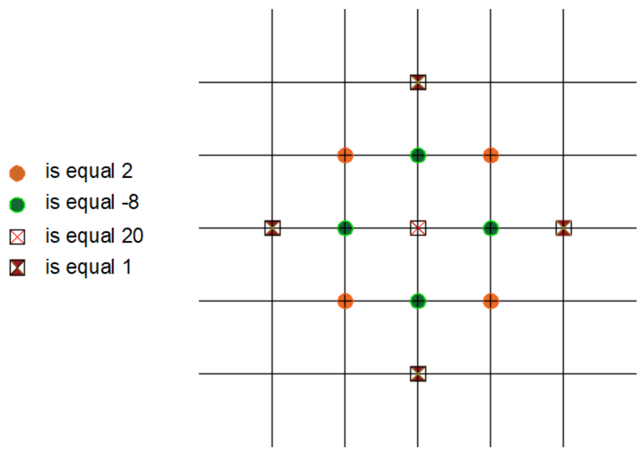

Figure (2): Biharmonic equation pattern for displacement at interior nodes.

METHODOLOGY

Twelve slabs are used to illustrate the proposed methods where each slab as its own dimensions and loading type, all slabs are simply supported at all four sides, the first slab is 6×6 m under $5KN/m^2$ uniform pressure load i.e. Slab 1, the properties of RC slab which are mentioned in Table (1) are constant parameters in all twelve slabs. Table (2) provide dimensions and loading types for all twelve slabs, to illustrate the accuracy and reliability of the Matlab Code, the each slab were analysed by all three methods.

Table (1): slab's properties

Modulus of elasticity	Gpa	21.7
thickness	mm	120
Poisson ratio	---	0.2

Table (2): slab dimensions and loading types

Name	Dimension m^2	load type
Slab 1	6×6	uniform pressure $5KN/m^2$
Slab 2	4×4	uniform pressure $5KN/m^2$
Slab 3	8×8	uniform pressure $5KN/m^2$
Slab 4	6×6	uniform pressure $10KN/m^2$
Slab 5	4×4	uniform pressure $10KN/m^2$
Slab 6	8×8	uniform pressure $10KN/m^2$
Slab 7	6×6	triangle load $5KN/m^2$ in x-direction and constant in y-direction
Slab 8	4×4	triangle load $5KN/m^2$ in x-direction and constant in y-direction
Slab 9	8×8	triangle load $5KN/m^2$ in x-direction and constant in y-direction
Slab 10	6×6	triangle load $10KN/m^2$ in x-direction and constant in y-direction
Slab 11	4×4	triangle load $10KN/m^2$ in x-direction and constant in y-direction
Slab 12	8×8	triangle load $10KN/m^2$ in x-direction and constant in y-direction

Each type of the slabs mentioned above was solved using the Code (FDM), Ansys (EEM), and Exact Solution (Analytical) where possible.

Boundary Discretization

For simple support boundary condition, the deflection and second derivative of deflection at the boundary is equal to zero, in other words, $z(x,y)=0$ and $\partial^2 z(x,y)=0$ at the external nodes, by using the same approximation derivative method for the conditions we have;

$$z(0,y)=z(a,y)=0 \tag{8}$$

$$z(x,0)=z(x,b)=0 \tag{9}$$

Where a & b are the length and width of the slab

$$\partial^2 z_i(0,y) \approx \frac{z_i-2z_{i+1}+z_{i+2}}{\Delta^2}=0 \tag{10}$$

$$\partial^2 z(a,y) \approx \frac{z_i-2z_{i-1}+z_{i-2}}{\Delta^2}=0 \tag{11}$$

$$\partial^2 z_j(x,0) \approx \frac{z_j-2z_{j+1}+z_{j+2}}{\Delta^2}=0 \tag{12}$$

$$\partial^2 z_i(x,b) \approx \frac{z_j-2z_{j-1}+z_{j-2}}{\Delta^2}=0 \tag{13}$$

Discretization of FDM

Programming and Discretization of the MATLAB code is presented as matrices so the deflection equation (3) can be written as a matrix as follows:

$$[A][Z]=[F] \tag{14}$$

Where:

$[Z]$ is the deflection matrix.

$[F]=[\frac{q(x,y)}{D}\times \Delta_x^2\Delta_y^2]$ is the lift hand side of the Deflection equation as a matrix

$[A]=[U_{xxxx}]+2[U_{xxyy}]+[U_{yyyy}]+[BC]$ is the Biharmonic Operator matrix

$[BC]$ is the boundary condition matrix.

First, a matrix provides the boundary conditions (BC) which satisfy equations form 8 to 13 , see figure (3).

Regarding the deflection equation (3), three matrices were developed, one for each term as follows U_{xxxx}, U_{xxyy}, and U_{yyyy} represent the equations (4), (6), and (5), these matrices are combined together to give a pattern scheme as shown in figure (2), adding the previous matrices with the boundary condition matrix (BC) provides the left hand side of the deflection equation.

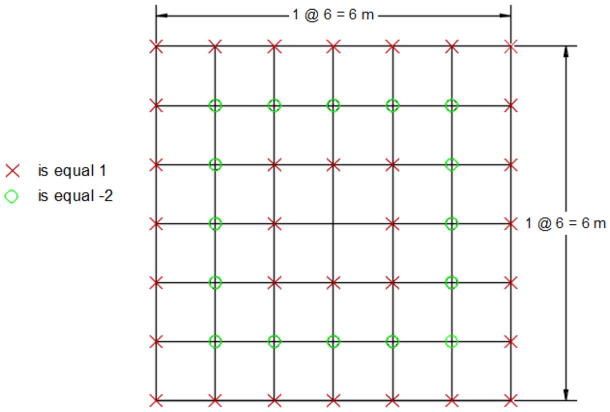

Figure (1): Boundary condition scheme for matrix (BC)

RESULTS AND DISCUSSIONS

By solving the matrices of the left hand side mentioned in the previous section along with the right hand side which consists of the Flexural rigidity (D) and the applying load $q(x, y)$ gives the deflection at any node in the slab. Figure (4) shows the deflection for Slab 1, where the maximum deflection at mid span was 7.8069 mm.

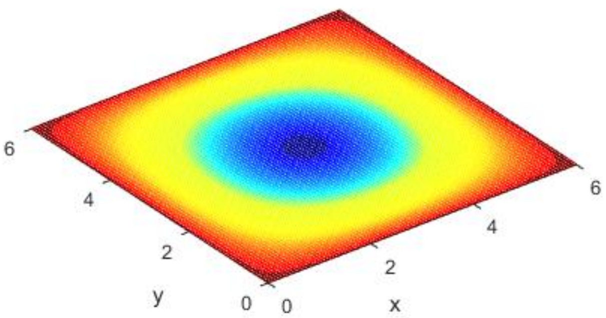

Figure (4): Deflection results for Slab 1 using FDM

The Finite Element Method (FEM) is a numerical technique used to solve partial differential equations in engineering and physics. In FEM, the domain of interest is divided into smaller, simpler elements, there are different types of mesh which can be used and they depends on the element shape (triangles, tetrahedrons, etc.) used in the analysis the Ansys is a software operate based on Finite Element Method (FEM), by modeling the Slab geometry, inputting the material properties, and choosing the right mesh settings, one can obtain the deflection at any point with high accuracy. Figure (5) shows the results for Slab 1 with maximum deflection equal to 8.2819 mm.

As mentioned previously, most PDEs have no exact solution and regarding of a 4th order PDEs, the analytical solution is even more challenging to obtain. However, exact solution for the Deflection Equation is provided for simple supported slab only since each type of support provides different boundary condition to PDEs. Thus, simple support condition is modeled as $z(x, y) = 0$ and $\partial^2 z(x, y) = 0$ at the supports, if the boundary conditions are satisfied, the analytical solution for the deflection is expressed in double trigonometric series form referred to as **Navier's method** [11] which is giving by

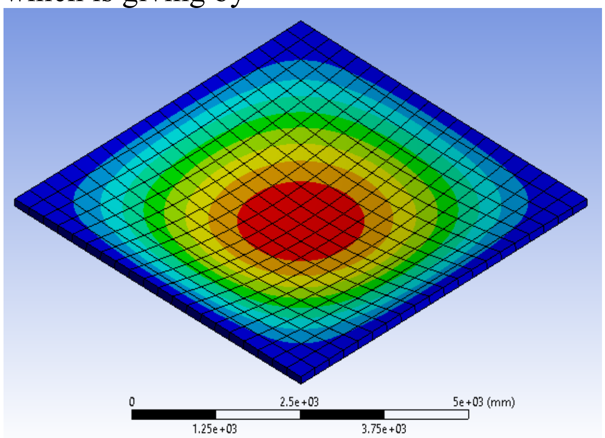

Figure (2): Deflection results for Slab 1 using FEM

$$z(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \qquad (15)$$

Where a & b are the dimensions of the plate, m & n are integers in the deflection calculation, they both equal to 1 so only the first term of Navier's series is adequate.

A_{mn} is constant which satisfy the boundary giving by:

$$A_{mn} = \frac{4q_{mn}}{Dab\pi^4 \left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}^2}$$

$$q_{mn} = \int_0^a \int_0^b q(x, y) \times \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy$$

In Slab 1, the load function is constant $(x, y) = 5\ KN/m^2 = q_0$, thus;

$$q_{mn} = \frac{4q_0 ab}{mn\pi^2} \quad for\ m, n = 1,3,5, \ldots$$

Hence,

$$A_{mn} = \frac{16q_0}{Dmn\pi^4 \left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}^2} \quad for\ m, n = 1, 3, 5, \ldots.$$

Then

$$z(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16q_0}{Dmn\pi^4 \left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}^2} sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right)$$

By applying Slab 1 parameters, the maximum deflection at mid-point was 8.2671 mm.

The results form Slab 1 to Slab 6 were less than the exact solution by 5.56 % as showing in table (3). Figures (6) and (7) demonstrates the deflection using all three methods for Slab 1 and Slab 4.

Table (3): Maximum deflection (mm) for Slab 1 to Slab 6 using the three methods,

Slab No	Analytical	Ansys	Matlab code	Ansys/ Analytical	Matlab/ Analy
Slab 1	8.2671	8.2819	7.8069	0.9981	0.943
Slab 2	1.6361	1.6546	1.5430	1.0113	0.943
Slab 3	26.1782	25.969	24.688	0.9920	0.943
Slab 4	16.5659	16.497	15.623	0.9958	0.943
Slab 5	3.2723	3.3092	3.0860	1.0113	0.943
Slab 6	52.3564	51.938	49.376	1.0081	0.943

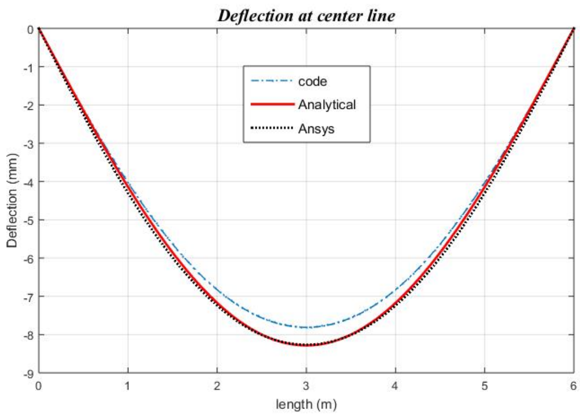

Figure (3): Slab 1 deflection using MATLAB code, Ansys, and Analytical solution.

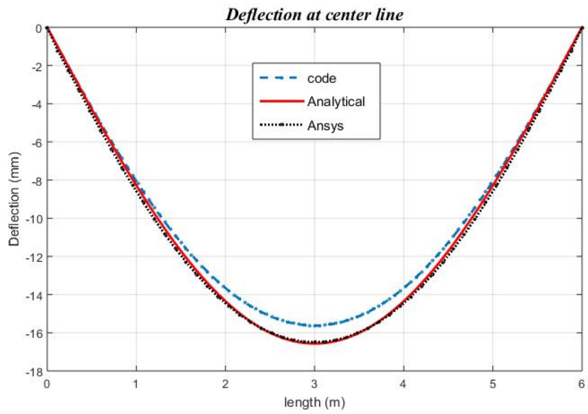

Figure (4): Slab 4 deflection using MATLAB code, Ansys, and Analytical solution.

As for slab 7 to Slab 12 were load type has changed from unform loads to triangle loads, the deflection values has slightly improved which provides better approximation to the exact solution, the FDM results were less than the exact solution by 4.66 %, see table (4). Figures (8) and (9) demonstrates the deflection using all three methods for Slab 7 and Slab 10.

Table (4): Maximum deflection (mm) for Slab 7 to Slab 12 using the three methods,

Slab No	Analytical	Ansys	Matlab code	Ansys/ Analytical	Matlab/ Analytical
Slab 7	4.1415	4.1705	3.9483	1.0070	0.953
Slab 8	0.8181	0.8334	0.7799	1.0187	0.953
Slab 9	13.0891	13.111	12.4786	1.0017	0.953
Slab 10	8.2829	8.3411	7.8966	1.0070	0.953
Slab 11	1.6361	1.6667	1.5598	1.0187	0.953
Slab 12	26.1782	26.221	24.9571	1.0016	0.953

CONCLUSION

In the paper, twelve simple support reinforced concrete slabs were model and analyzed for deflection using three different methods. One analytical method and the other two method are numerical methods. The analytical solution is obtain by applying Navier's method approach, the second method is the Finite Element Method (FEM) which is adopted in Ansys software and the third method is the Finite Different Method The FDM employed Matlab code. Half of the slabs were under uniform pressure with different loads and the other half were under triangle load in x-direction and constant in y-direction with different values (5 KN/m^2 &10 KN/m^2). In addition, All slabs dimensions vary between 4x4, 6x6 and 8x8. The following conclusions can be made:

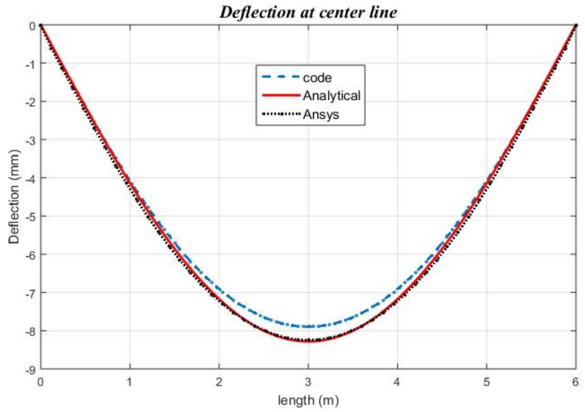

Figure (5): Slab 7 deflection using MATLAB code, Ansys, and Analytical solution

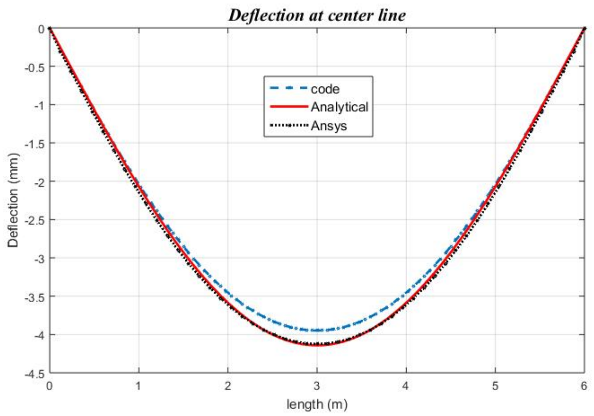

Figure (6): Slab 10 deflection using MATLAB code, Ansys, and Analytical solution

1.The developed FDM code results were about 5.57 % less than the exact solution for slabs with uniform loads and about 4.67 % for slabs with triangle loads thereby, the developed code is more sufficient in triangle load than uniform load.

2. Changes in the slab's dimensions did not yield improvements or deteriorate the solution obtained using the Finite Difference Method.

3. The FEM results was about 0.49% from the analytical solution results based on average values for all slabs.

4. Coincidently, the analytical solution for slab deflection PDE is not difficult to obtain for simple support slabs which can be evaluated by using Navier's method, to the best of the author's knowledge there is only two methods to solve the slab's deflection equation with simple support condition analytically namely, the Navier's method and the levy's solution. However, for slabs with fixed ends and free ends analytical is not possible.

In regard of stability and readability for the Matlab developed code, beast on the results presented here which showed a Variation about 5 % , the developed code is acceptable and showed good conversion to the exact solution, it's recommended to increase the deflection obtained by the MATLAB code by 5% to improve the calculated results. It should be noted that number of divisions ($\Delta_x = \Delta_y$) was 100 in all slabs, in other words number of nodes in each line was a hundred node so by increasing the number of nodes better results and less error can be obtain.

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